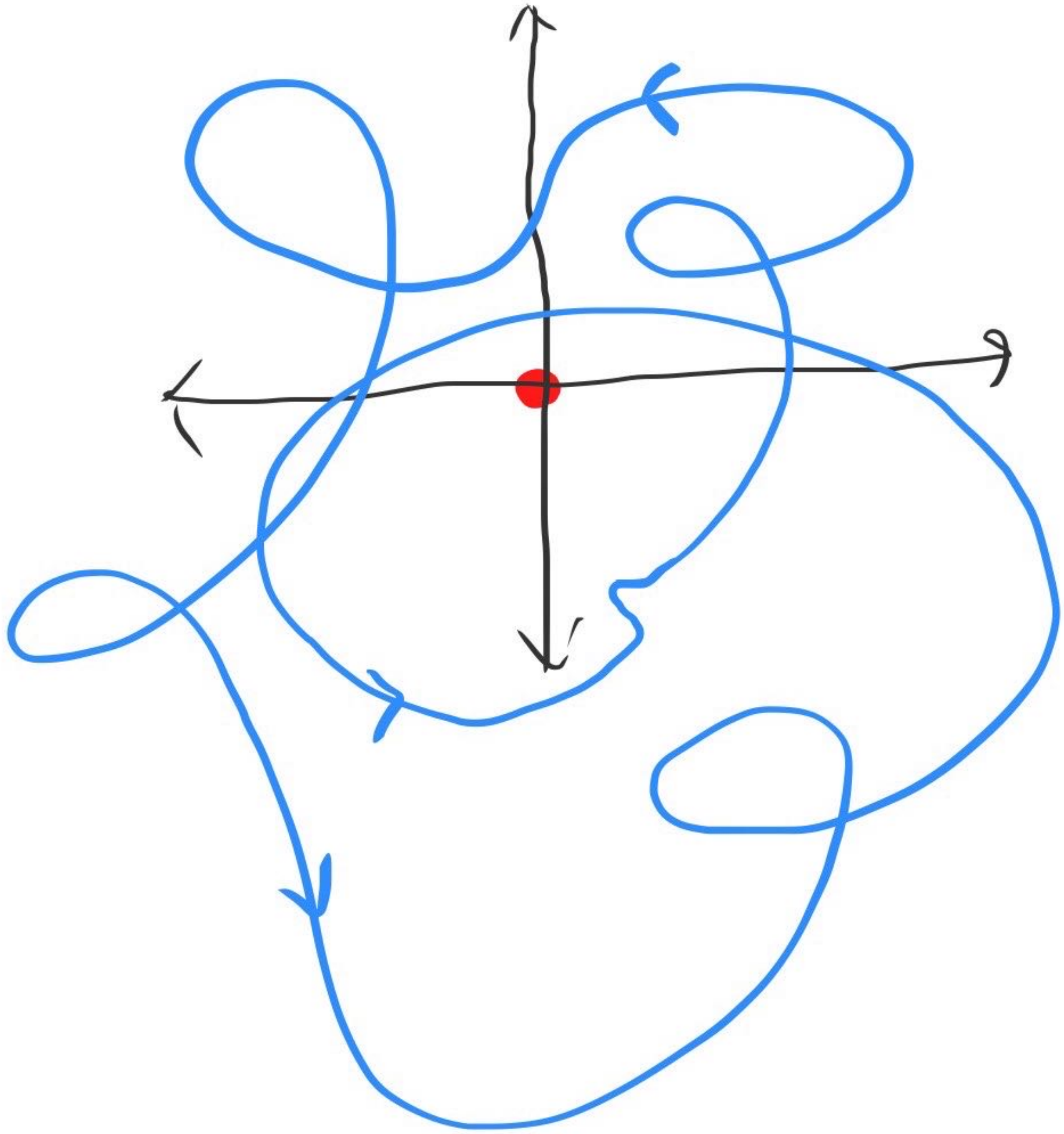


The winding number

(Suggested reference: "Winding around" - John Roe)

Any ~~good~~ great idea comes in many forms.

"If $\gamma: S^1 \to \mathbb{R}^2$ is a loop not passing through 0, there is a well-defined integer $wn(\gamma) \in \mathbb{Z}$

If γ_1 is deformable to γ_2, not passing thru 0 ("homotopic" as maps to $\mathbb{R}^2 - 0$) then $wn(\gamma_1) = wn(\gamma_2)$.

1. Complex analysis.

For γ piecewise smooth ...

$$\mathrm{wn}(\gamma) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$$

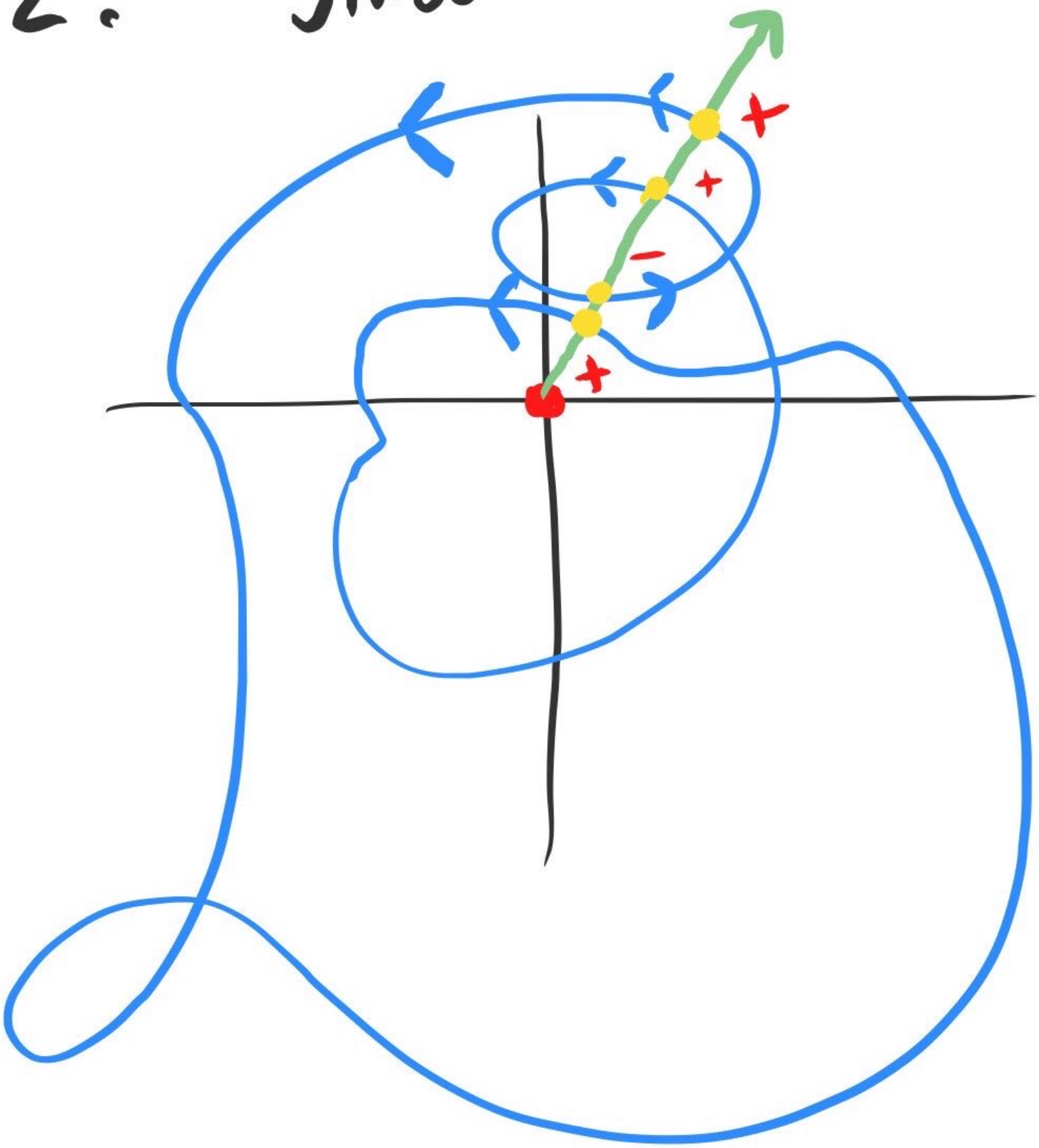
$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)}\, dt$$

1. The heck does this **mean**?
2. $\mathrm{wn}(\gamma) \in \mathbb{Z}$?
3. $\gamma_1 \sim \gamma_2 \Rightarrow \mathrm{wn}(\gamma_1) = \mathrm{wn}(\gamma_2)$

(Green's theorem).

Why this way? Residue thm!

2. Smooth manifolds



$\mathrm{wn}(\gamma)$

$= \#(r_\theta \cap \gamma)$

counted w/ sign

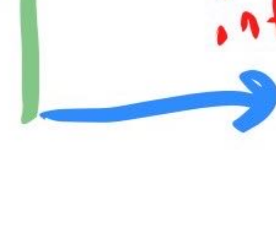
(add 1 for +ve subtract for −ve)

$3-1=2$

Choose a <u>generic</u> angle θ

At an intersection point of $r_\theta \cap \gamma$

+ve intersection



−ve intersection

Downside

1) Why does this not depend on Θ?

2) Why does $\gamma_1 \sim \gamma_2$

$$\Rightarrow wn(\gamma_1) \cap wn(\gamma_2)$$

Upsides

1) Generalizes wildly

2) Neat trick: immediately tell inside & outside of simple

3) Algebraic topology / covering spaces

If $\eta : S^1 \to \mathbb{R}^2$ misses 0...

radially project to get

a map $\gamma : S^1 \to S^1$

(this records angle η is at, not radius).

Think of γ as a map from

$[0, 2\pi]$ w/ $\gamma(0) = \gamma(2\pi)$

Helix H

Fact: a map $\gamma : [0, 2\pi] \to S^1$ has a lift $\tilde{\gamma} : [0, 2\pi] \to H$

H records angle...
it also records # of times you winded around.

Def: $wn(\gamma) =$ # of levels $\tilde{\gamma}(2\pi)$ is above $\tilde{\gamma}(0)$.

Upshot:

1) Comprehensible visually
2) $wn(\gamma) \in \mathbb{Z}$
3) Leads to covering space thm.

Downside:
Technical work required to establish lift.

Applications in...

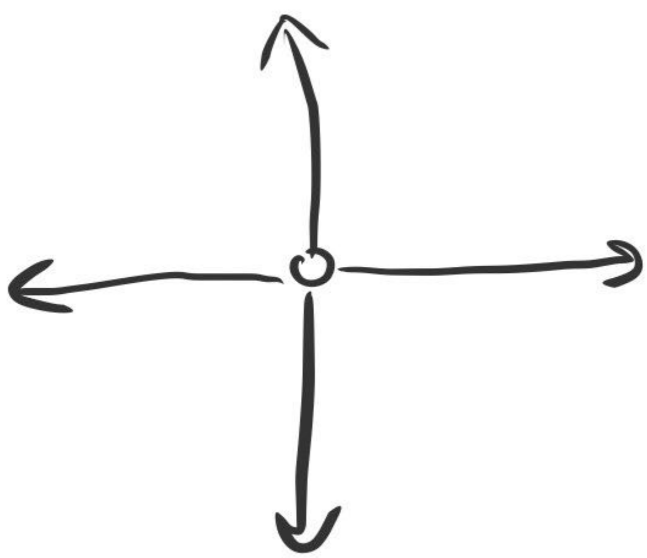
1) Algebra
2) Analysis
3) Topology

Thm: Let $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ is a nonconstant polynomial w/ complex coefficients.

Then $\exists\, z \in \mathbb{C}$ s.t. $p(z) = 0$.

(Fundamental theorem of algebra)

Pf: Suppose, towards a contr. $p(z) \neq 0$ for any z.



Consider

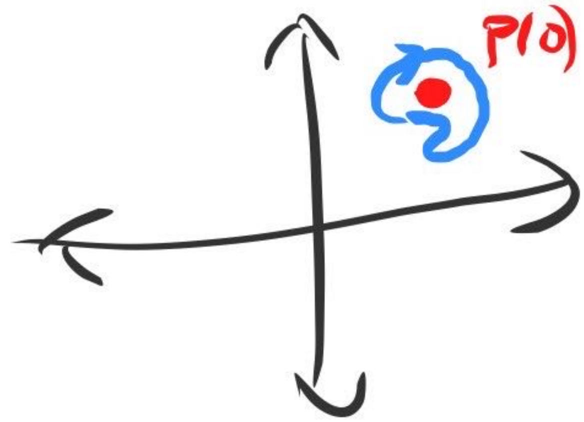
$$P(S^1_r)$$

circle of radius r

1) $\mathrm{wn}(p(S'_\varepsilon)) = 0$

(uses $p(z) \neq 0$)

Pf:



C'mon, this doesn't wind.

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2) $\mathrm{wn}(p(S'_r))$ indep of r

(uses $p(z) \neq 0$)

b/c varying r deforms one $p(S'_r)$ to $p(S'_R)$ w/o passing thru 0.

3) For $R \gg 0$ (really big)

then $\operatorname{wn}(p(S_R^1)) = \deg(p)$

This will be our contr. !

Pf of #3

Consider $P_t(z) =$

$$z^n + t a_1 z^{n-1} + \cdots + t a_n$$

$$P_0(z) = z^n$$
$$P_1(z) = p(z)$$

If $P_t(z) \neq 0$ for any $z \in S^1_R$ then

$P_t(S^1_R)$ is a deformation from $p(S^1_R)$ to $P_0(S^1_R) = S^1_R{}^n$ traversed n times.

$wn = n.$

Well...

$$P_t(z) = z^n + t a_1 z^{n-1} + \cdots + t a_n$$

$$|P_t(z)| \geq |z|^n - \underbrace{t|a_1 z^{n-1} + \cdots + a_n|}$$

$$t|a_1 z^{n-1} + \cdots + a_n|$$

$$\leq \left(|a_1||z|^{n-1} + \cdots + |a_n|\right)$$

$$\leq R^{n-1}\left(|a_1| + \cdots + |a_n|\right)$$

$$\geq R^{n-1}\left(R - |a_1| - \cdots - |a_n|\right)$$

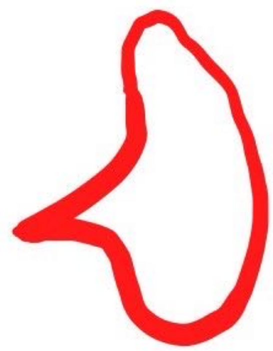
Take $R > |a_1| + \cdots + |a_n|$.

In topology...

A **regular** loop $\gamma : S^1 \to \mathbb{R}^2$

is one s.t. $\gamma'(t) \neq 0$

for any t.

(Not like )

Two are **regularly homotopic** if

I can deform γ_1 to γ_2 thru

regular loops.

Thm: You can't turn the circle inside out.

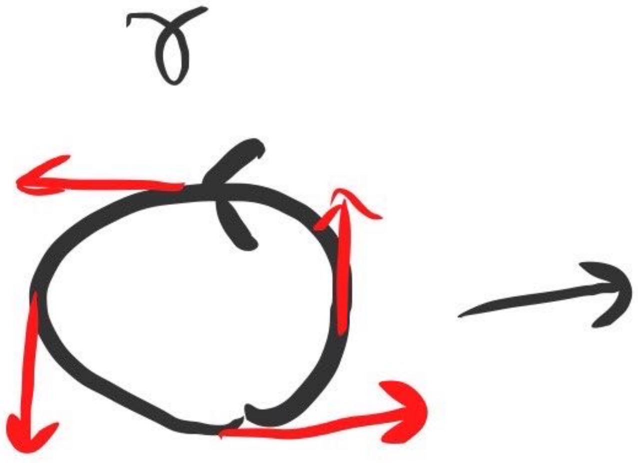
AKA: $\gamma_1(t) = (\cos t, \sin t)$ is not regularly homotopic to

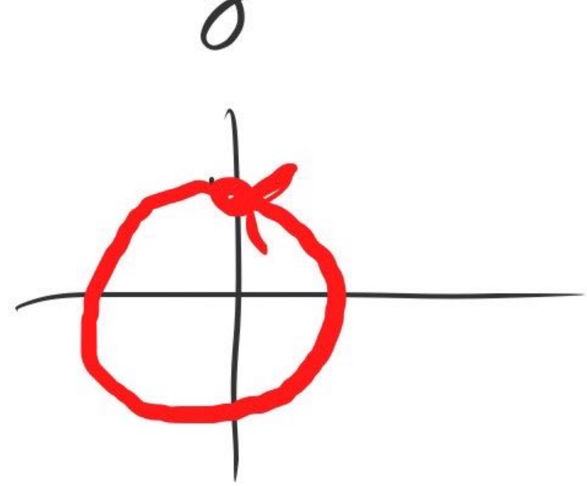
$$\gamma_2(t) = (\cos t, -\sin t)$$

γ' is never 0 — so gives a loop in $\mathbb{R}^2 \setminus 0$.

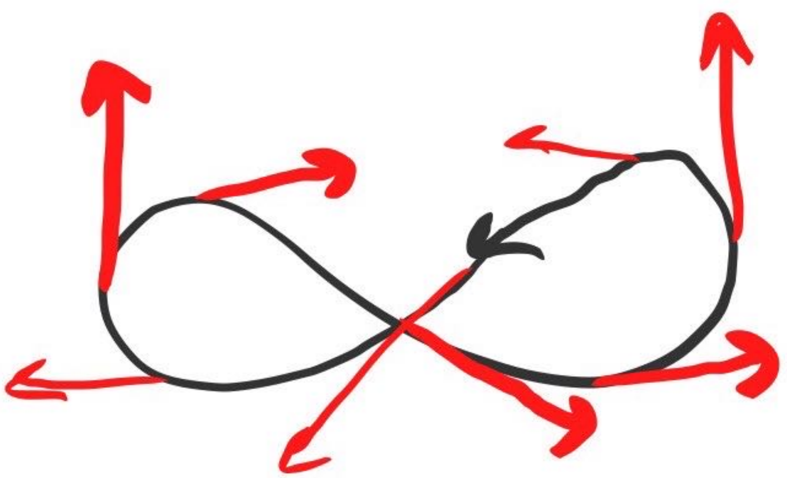
Say the "turning number" is $wn(\gamma')$.

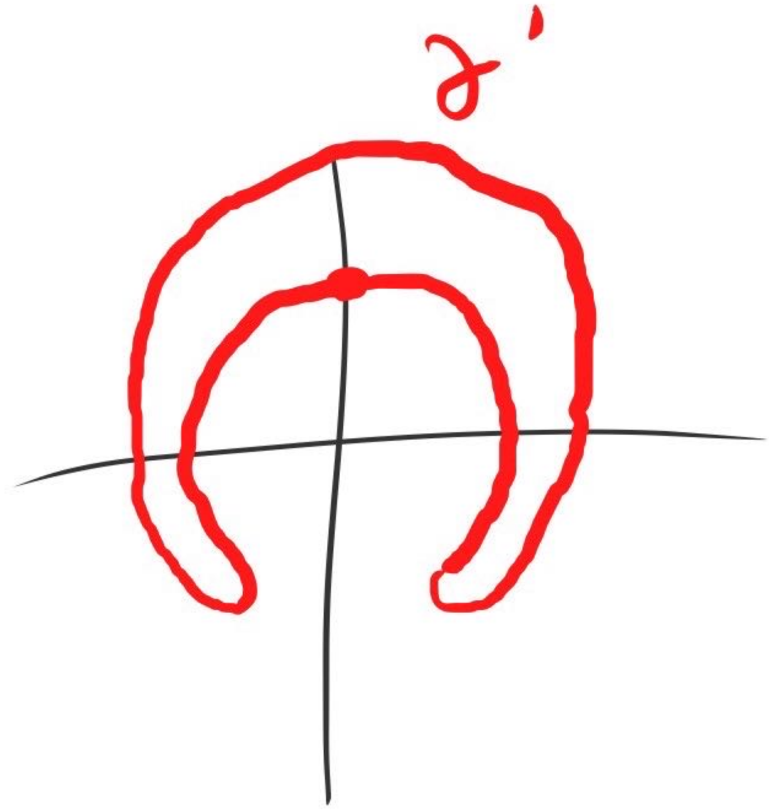
$tn(\gamma_1) = 1$ $\quad$ $tn(\gamma_2) = -1$





tn = 1





↳ Turning number 0.

Fact: you <u>can</u> turn S^2 inside out. (See "Outside in" on youtube)

Thm (Smale-Hirsch)

S^n can be turned inside out

iff $n = 0, 2, 6$